Uniformly travelling water waves from a dynamical systems viewpoint: some insights into bifurcations from Stokes' family

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Numerical work of many people on the bifurcations of uniformly travelling water waves (two-dimensional irrotational gravity waves on inviscid fluid of infinite depth) suggests that uniformly travelling water waves have a reversible Hamiltonian formulation, where the role of time is played by horizontal position in the wave frame. In this paper such a formulation is presented. Based on this viewpoint, some insights are given into bifurcations from Stokes' family of periodic waves. It is demonstrated numerically that there is a 'fold point' at amplitude $A_0 \approx 0.40222$. Assuming non-degeneracy of the fold and existence of an associated centre manifold, this explains why a sequence of p/q-bifurcations occurs on one side of A_0, with $0 < p/q \leqslant \frac{1}{2}$, in the order of the rationals. Secondly, it explains why no symmetry-breaking bifurcation is observed at A_0, contrary to the expectations of some. Thirdly, it explains why the bifurcation tree for periodic uniformly travelling waves looks so much like that for the area-preserving Hénon map. Fourthly, it leads to predictions of a rich variety of spatially quasi-periodic, heteroclinic and chaotic waves.

1. Introduction

The study of waves propagating on the surface of water has attracted a lot of attention from the scientific community for more than a century, in particular the study of uniformly travelling waves, those which are steady in a horizontally moving Galilean frame. A question of considerable activity concerns the classification of such surface waves and their bifurcations.

In this paper we concentrate on gravity waves of permanent form on the free surface of an ideal fluid (incompressible and inviscid) of infinite depth with no surface tension. We suppose that the flow is two-dimensional, irrotational and in a vertical plane.

Many interesting results have been found for this problem. In particular, there is a family of periodic waves, with one trough and crest per period, which connects the infinitesimal sinusoidal waves of linear theory with waves with a 120° crest (Amick, Fraenkel & Toland 1982). Numerically, this seems to be a smooth family (e.g. Longuet-Higgins 1985; Longuet-Higgins & Fox 1978), with two parameters, suitable ones being the wavelength λ and the trough to crest height $2a$, though because of a scaling symmetry only the combination $A = ak$ is important (where $k = 2\pi/\lambda$). The parameter A runs from 0 at the flat state to $A_{\max} \approx 0.4434$ for the wave with the 120° (Longuet-Higgins 1984a). We call it *Stokes' family*, after his pioneering work on the problem.

Bifurcations of other periodic uniformly travelling waves with more than one crest and trough per period, from Stokes' family, have been observed numerically, for A in the range $(A_0, A_{\frac{1}{2}}]$, where $A_0 \approx 0.402233$, $A_{\frac{1}{2}} \approx 0.404961$ (e.g. Chen & Saffman 1980; Saffman 1980; Longuet-Higgins 1985, 1986; Zufiria 1987b; Aston 1991). These bifurcations can be characterized by a coprime pair of integers (p, q), $1 \leqslant p < q$. The difference between the branching wave and the Stokes wave has the form $e^{\pm ipkx/q} P(x, y)$ to first order, with $P(x + \lambda, y) = P(x, y)$. These numerical results suggest there is a dense set of bifurcation points $A_{p/q}$ of type (p, q) in the interval $(A_0, A_{\frac{1}{2}}]$, in the same order as the rationals p/q in $(0, \frac{1}{2}]$. Furthermore, secondary bifurcations of periodic waves are observed from many of these new branches, and in fact an apparently infinitely branching bifurcation tree. On the other hand, no bifurcations were observed for $0 < A < A_0$, and despite searching, no bifurcation was found at A_0 (Chen & Saffman 1980; Longuet-Higgins 1985; Zufiria 1987b). This was found strange by some authors, as superharmonic bifurcation (i.e. generating solutions of the same period) breaking the reflectional symmetry might have been expected (though Longuet-Higgins (1986) did not hold that view).

The above numerical results are strongly reminiscent of the generic bifurcations of periodic orbits for reversible Hamiltonian systems of two degrees of freedom and the related reversible area-preserving mapping, such as the Hénon map (Hénon 1969). It is the aim of our paper to explain the connection.

We do this by finding a reversible Hamiltonian formulation for uniformly travelling water waves, and demonstrating numerically the existence of a fold in Stokes' family. Under the following two assumptions:

1. the fold has a finite-dimensional centre manifold on which the Hamiltonian formulation defines a smooth flow;
2. the fold satisfies the non-degeneracy conditions of Rimmer (1983) (which are generic for smooth reversible Hamiltonian systems);

we deduce the existence of a bifurcation scenario very similar to that for the Hénon map. In particular there is a dense interval of bifurcation on one side of A_0 and no symmetry-breaking bifurcation at A_0. Furthermore, this point of view leads to many predictions about the occurrence of other sorts of uniformly travelling water waves, such as quasi-periodic, heteroclinic and chaotic waves. The paper concludes with some conjectures about the almost highest waves, and comments.

2. Hamiltonian formulation

We denote the velocity of a uniformly travelling irrotational wave by c (relative to fluid at infinite depth), the surface elevation by $\eta(x)$, $-\infty < x < \infty$, and the velocity potential and stream function (in the frame of the wave) by $\phi(x, y)$ and $\psi(x, y)$ respectively, $y \leqslant \eta(x)$, $-\infty < x < \infty$; here x is the horizontal coordinate in the frame of the wave and y is the vertical coordinate. The velocity field (u, v) is given by $u = \phi_x = \psi_y, v = \phi_y = -\psi_x$ and satisfies Laplace's equation:

$$\Delta\psi = \Delta\phi = 0, \quad -\infty < x < \infty, -\infty < y \leqslant \eta(x),$$

with the following boundary conditions: at the bottom

$$(u, v) \to (-c, 0) \quad \text{as} \quad y \to -\infty;$$

on the surface $y = \eta(x)$

$$\psi = \text{constant, equivalently } u\eta_x = v \text{ (kinematic condition)},$$

$$\tfrac{1}{2}(u^2 + v^2) + g\eta = K, \text{ constant (Bernoulli condition)},$$

where g is the acceleration due to gravity. We choose the timescale so that $g = 1$, and the origin of y so that $K = \frac{1}{2}c^2$ (then the mean level, when defined, is zero). Without loss of generality, $\psi = 0$ on the surface.

A Hamiltonian formulation for the time evolution of the whole wavefield was found by Zakharov (1968). We do not consider temporal stability of uniformly travelling waves. Nonetheless, we take a dynamical viewpoint: we will regard the equations for uniformly travelling water waves as a dynamical system for quantities defined on a vertical line, where the role of time is played by the horizontal coordinate x.

Formally, the transformation to a dynamical system in x is straightforward. For example, let M be the space of triples (η, ψ, v) such that $\eta \in \mathbb{R}$, ψ and v are functions from $(-\infty, \eta] \to \mathbb{R}$, $\psi_y \to -c$ and $v \to 0$ as $y \to -\infty$, $\psi(\eta) = 0$ and $\frac{1}{2}(v^2 + \psi_y^2) + \eta = \frac{1}{2}c^2$ at $y = \eta$. Then the equations for uniformly travelling water waves can be rewritten as the following evolution equations on M:

$$\eta_x = v/\psi_y, \quad \psi_x = -v, \quad v_x = \psi_{yy} \quad \text{for} \quad y \in (-\infty, \eta).$$

A similar formulation can be made with ϕ and u as the dependent variables. This will turn out to have the structure of a reversible Hamiltonian system.

We recall the general definition of a reversible Hamiltonian system. A *symplectic manifold* is a manifold M with a closed non-degenerate† 2-form ω, called the *symplectic form*. Given a function $H : M \to \mathbb{R}$, called a *Hamiltonian*, the symplectic form induces a vector field X_H on M, the unique one such that

$$\omega(\xi, X_H) = \mathrm{d}H(\xi), \forall \xi. \tag{1}$$

It is called a *Hamiltonian vector field*. If there exists an anti-symplectic involution $S : M \to M$ (i.e. S^2 is the identity and S reverses the sign of ω) such that $H \circ S = H$, the Hamiltonian vector field is said to be *reversible*, because then $X_H \circ S = -DS \cdot X_H$. So to demonstrate a reversible Hamiltonian formulation for a problem, we need to specify M, ω, H and S, such that $\dot{z} = X_H(z)$ is equivalent to the equations for the problem.

We conjectured such a formulation for uniformly travelling water waves back in 1985, and some of the consequences of this conjecture formed the basis for the work of Zufiria (1987*a*, *b*). Recently, Mielke (1991) found a reversible Hamiltonian formulation for the case with surface tension and finite depth. Although the limits are singular, it turns out not to be hard to extend his approach to infinite depth and/or zero surface tension. We will treat now the cases of infinite depth with and without surface tension.

It is well known (e.g. Whitham 1974) that uniformly travelling water waves with surface tension κ and speed c on infinite depth, with Bernoulli constant $\frac{1}{2}c^2$, can be derived from Luke's variational principle $\delta \int L(\phi, \phi_x, \eta, \eta_x)\,\mathrm{d}x = 0$, where

$$L(\phi, \phi_x, \eta, \eta_x) = \frac{1}{2}\eta^2 + \kappa[(1 + \eta_x^2)^{\frac{1}{2}} - 1] + \int_{-\infty}^{\eta} \frac{1}{2}(\phi_x^2 + \phi_y^2 - c^2)\,\mathrm{d}y.$$

When $\kappa \neq 0$, one can Legendre transform and write this in Hamiltonian form. It is convenient first to standardize the domain for the function ϕ, and furthermore to replace it by the rest frame velocity potential. So let $Y = y - \eta(x)$ be the vertical height below the surface, and

$$F(x, Y) = \phi[x, \eta(x) + Y] + cx.$$

† ω non-degenerate means that $\omega(\xi, \eta) = 0 \forall \xi$ implies $\eta = 0$.

In terms of F and η, the Lagrangian has the form:

$$L(F, F_x, \eta, \eta_x) = \tfrac{1}{2}\eta^2 + \kappa[(1+\eta_x^2)^{\frac{1}{2}} - 1] + \int_{-\infty}^{0} \tfrac{1}{2}[(F_x - \eta_x F_Y - c)^2 + F_Y^2 - c^2]\,\mathrm{d}Y.$$

We obtain the conjugate variables

$$U = \delta L/\delta F_x = F_x - \eta_x F_Y - c,$$

$$w = \partial L/\partial \eta_x = \kappa\eta_x/(1+\eta_x^2)^{\frac{1}{2}} - \int_{-\infty}^{0} U F_Y\,\mathrm{d}Y.$$

The variable U has the interpretation

$$U(x, Y) = u(x, \eta(x) + Y),$$

and $w(x)$ has the interpretation of the sum of the vertical component of the surface tension force at $(x, \eta(x))$ and the flux of vertical momentum across the vertical line through x. The above relations can be inverted, yielding

$$F_x = \eta_x F_Y + U + c,$$

$$\eta_x = w'/(\kappa^2 - w'^2)^{\frac{1}{2}},$$

where

$$w' = w + \int_{-\infty}^{0} U F_Y\,\mathrm{d}Y.$$

Taking the Legendre transform, we obtain the Hamiltonian

$$H(F, U, \eta, w) = \kappa - (\kappa^2 - w'^2)^{\frac{1}{2}} - \tfrac{1}{2}\eta^2 + \int_{-\infty}^{0} \tfrac{1}{2}((U+c)^2 - F_Y^2)\,\mathrm{d}Y, \qquad (2a)$$

which can be recognized as Benjamin's flow-force invariant $\bar{S}$, a quantity which is already known to be conserved for uniformly travelling waves (Benjamin 1984). The phase space M for this Hamiltonian system consists of all quadruples (F, U, η, w), where $F, U: (-\infty, 0] \to \mathbb{R}$, $F(Y)$, $F_Y(Y) \to 0$, $U(Y) \to -c$ as $Y \to -\infty$, and $\eta, w \in \mathbb{R}$, with $w' \in (-\kappa, \kappa)$ satisfying the constraint

$$U_0 w'/(\kappa^2 - w'^2)^{\frac{1}{2}} = F_{Y0}, \qquad (2b)$$

where subscript 0 here and hereafter denotes the value at the surface $Y = 0$. The symplectic form is the restriction to M of the standard one

$$\omega = \mathrm{d}w \wedge \mathrm{d}\eta + \int_{-\infty}^{0} \mathrm{d}U \wedge \mathrm{d}F\,\mathrm{d}Y,$$

which is non-degenerate on M, because the constraint can be regarded as fixing F_{Y0} which does not enter ω. The equations of motion come out in the form

$$\begin{aligned} \eta_x &= \partial H/\partial w = w'/(\kappa^2 - w'^2)^{\frac{1}{2}}, \\ w_x &= -\partial H/\partial \eta = \eta, \\ F_x &= \delta H/\delta U = \eta_x F_Y + U + c, \\ U_x &= -\delta H/\delta F = \eta_x U_Y - F_{YY}, \end{aligned} \qquad (3)$$

which can be seen to be equivalent to the usual ones. The equation of motion for w expresses vertical momentum balance, and is equivalent to the Bernoulli condition.

The fourth equation is equivalent to Laplace's equation in the bulk, and using the first equation, the constraint (2b) is equivalent to the kinematic condition. The constraint (2b) is necessary to eliminate a delta-function surface contribution which would otherwise appear in the last equation. We are grateful to Tom Bridges for pointing out that we needed to deal with this and for suggesting the above solution.

Furthermore, the system is reversible, with respect to the following involution

$$S(F, U, \eta, w) = (-F, U, \eta, -w).$$

The limit $\kappa \to 0$ is singular: although the variational principle from which we started is good, the Legendre transform cannot be carried out. Nevertheless, a Hamiltonian formulation can still be found. We choose as Hamiltonian

$$H(F, U, \eta, w) = -\tfrac{1}{2}\eta^2 + \int_{\infty}^{0} \tfrac{1}{2}((U+c)^2 - F_Y^2)\, \mathrm{d}Y, \tag{4}$$

as phase space the manifold M' defined by replacing the constraint (2b) by the following two constraints:

$$w = -\int_{-\infty}^{0} U F_Y\, \mathrm{d}Y,$$

$$\eta + \tfrac{1}{2}(U^2 + F_Y^2)_0 = \tfrac{1}{2}c^2,$$

and the same symplectic form ω. It can be checked that ω restricted to M' is still non-degenerate, provided $U_0 \neq 0$ (this will come out automatically from the following calculation too). We now calculate the resulting equations of motion from the general definition (1) of a Hamiltonian vector field.

The constraints imply that associated to any perturbation $(\delta U, \delta F)$ in (U, F), we have perturbations

$$\delta\eta = -U_0\, \delta U_0 - F_{Y0}\, \delta F_{Y0},$$

$$\delta w = \int_{-\infty}^{0} (U_Y\, \delta F - F_Y\, \delta U)\, \mathrm{d}Y - U_0\, \delta F_0,$$

in η and w. Then taking a pair of such perturbations $(\delta U, \delta F)$ and $(\dot U, \dot F)$,

$$\omega((\delta U, \delta F), (\dot U, \dot F)) = \delta w\, \dot\eta - \dot w\, \delta\eta + \int_{-\infty}^{0} \delta U\, \dot F - \dot U\, \delta F\, \mathrm{d}Y$$

$$= \dot\eta \left\{ \int_{-\infty}^{0} (U_Y\, \delta F - F_Y\, \delta U)\, \mathrm{d}Y - U_0\, \delta F_0 \right\} + \dot w\{U_0\, \delta U_0 + F_{Y0}\, \delta F_{Y0}\} + \int_{-\infty}^{0} \delta U\, \dot F - \dot U\, \delta F\, \mathrm{d}Y.$$

Also

$$\mathrm{d}H(\delta U, \delta F) = -\eta\, \delta\eta + \int_{-\infty}^{0} (U+c)\, \delta U - F_Y\, \delta F_Y\, \mathrm{d}Y$$

$$= \eta\{U_0\, \delta U_0 + F_{Y0}\, \delta F_{Y0}\} + \int_{-\infty}^{0} ((U+c)\, \delta U + F_{YY}\, \delta F)\, \mathrm{d}Y - F_{Y0}\, \delta F_0.$$

The equation $\omega((\delta U, \delta F), (\dot U, \dot F)) = \mathrm{d}H(\delta U, \delta F)$, $\forall (\delta U, \delta F)$, can be seen to have a unique solution for $U_0 \neq 0$, namely

$$\dot F = \dot\eta F_Y + U + c,$$

$$\dot U = \dot\eta U_Y - F_{YY},$$

$$\dot\eta = F_{Y0}/U_0,$$

$$\dot w = \eta.$$

These correspond to the correct equations for uniformly travelling waves with no surface tension, with $\dot{\eta}$ interpreted as η_x, and so on. Thus our goal is achieved.

It should be possible to use the same technique to obtain a Hamiltonian formulation for water waves with no surface tension on finite depth.

3. Well-posedness

A problem with all dynamical systems formulations of elliptic problems is that the resulting evolution equations are ill-posed. The essence of the problem is that the solution of Laplace's equation $\Delta\phi = 0$ with (ϕ, ϕ_x) given on $x = 0$, $\phi = 0$ on $y = 0$ and 1, while existing and unique, is not continuous with respect to the initial data (ϕ, ϕ_x) on $x = 0$. However, in many examples a finite-dimensional submanifold on which the evolution equations are well-posed can be found, e.g. Benjamin (1967) for the Benjamin–Ono equation, and Kirchgässner (1982) using centre manifold theory. A *centre manifold* is a local invariant manifold in the neighbourhood of some particular solution, tangent to the set of neutral infinitesimal perturbations (where the definition of neutral depends on the context). This idea has been applied to many similar problems now (see the surveys of Kirchgässner (1988) and Mielke (1991); also Amick & Kirchgässner 1989; Amick & Turner 1989).

Two major problems remain with the centre manifold approach, nonetheless. The first is that current proofs do not apply to the case of infinite depth, though whether there is a genuine obstruction is an open question. The second is that applications so far have been limited to a neighbourhood of a flat state. This is not a fundamental problem, however. In finite-dimensional Hamiltonian systems, if one finds, for example, a periodic solution with a normal Floquet multiplier $+1$ (see §4 for a definition), there exists an associated centre manifold, tangent to the subspace of perturbations corresponding to the Floquet multiplier $+1$. We are going to assume the same holds in our case, and that our Hamiltonian formulation defines a smooth flow on such a centre manifold (assumption 1 of §1).

Although we will not require it in this paper, in fact we conjecture that a much stronger property frequently holds, namely that for many of these ill-posed evolution problems there exists a finite-dimensional manifold in the phase space, on which the evolution equations define a smooth flow and which contains all the bounded solutions of the problem. Any such manifold of minimal dimension we propose to call an essential manifold. In the Hamiltonian context, we also require the essential manifold to be a symplectic submanifold, so that the flow on it is still Hamiltonian. The idea is a conservative analogue of inertial manifolds, which are finite-dimensional manifolds which attract exponentially all solutions of a dissipative semi-flow. Existence of an essential manifold for some elliptic systems has recently been proved by Mielke (1990). Although we suspect that an essential manifold does not strictly speaking exist for the uniformly travelling water-wave problem, because it looks as if its dimension would have to be infinite to include any waves with a 120° crest (see §7), we believe that one does exist if any neighbourhood of all such waves is excluded.

4. Dynamical systems interpretation of bifurcation results

We have given a Hamiltonian formulation for uniformly travelling water waves as evolution equations in x. The initial-value problem may not be well-posed, but under assumption 1 (§1), if we find a fold in the set of periodic solutions then there will be a finite-dimensional centre manifold on which they define a smooth flow.

It is important to be clear about what is meant by a fold in the set of periodic solutions. For an autonomous Hamiltonian system, it is a periodic solution for which there exist arbitrarily small deformations which are also periodic solutions, of close but not necessarily the same period, but not just a phase shift, which have the same energy to first order. We will see in the next section how to interpret this in the context of water waves.

An alternative description of a fold can be given in terms of the *Floquet multipliers* of a periodic solution. These are the complex numbers α such that there exists a solution of the linearized equations about the given periodic one which is also an eigenvector of the operator of translation by one period with eigenvalue α. Since for a periodic solution of an autonomous Hamiltonian system there is always a Floquet multiplier $+1$ corresponding to phase shift, and its generalized eigenspace usually contains another vector corresponding to continuation of the solution as the energy changes, one defines the *normal Floquet multipliers* to be the solutions of the above problem restricted to a given energy, and identifying any two perturbations which differ only by a phase shift. Then a fold can be characterized as a periodic solution with a normal Floquet multiplier $+1$. For a Hamiltonian system the Floquet multipliers come in reciprocal pairs $(\alpha, \alpha^{-1}), \alpha \in \mathbb{C}$, and $+1$ is a multiplier of even (algebraic) multiplicity $\geqslant 2$. For symmetric solutions of reversible systems this follows alternatively from reversibility. It is convenient to take account of this fact by defining the residues of a periodic orbit to be the quantities

$$R = \tfrac{1}{4}(2 - \alpha - \alpha^{-1}),$$

corresponding to each pair of normal Floquet multipliers.

Generically (meaning if a certain finite number of combinations of Taylor coefficients are non-zero), the multiplicity of the Floquet multiplier $+1$ at a symmetric fold of a finite-dimensional reversible Hamiltonian system is 4, so there is a four-dimensional centre manifold, and one of the following three results hold (depending on the signs of the combinations of Taylor coefficients; Rimmer 1983).

SC (saddle-centre bifurcation) (usually, but misleadingly in the Hamiltonian context, called a saddle-node bifurcation). The fold solution is contained in a one-parameter family S_t of symmetric periodic solutions (with $dS_t/dt \neq 0$), along which $H \sim bt^2$, some $b \neq 0$, and the (only) residue $R \sim at$, some $a \neq 0$ (where $t = 0$ corresponds to the fold and without loss of generality the energy of the fold is chosen to be zero), and there are no other nearby periodic solutions which are small deformations of the fold solution.

SB1 (supercritical symmetry breaking). The fold solution is the intersection of two one-parameter families of periodic solutions, one of which, S_t, consists of symmetric solutions and satisfies $H \sim ct$, $R \sim dt$, and the other, A_t, forms a pair of asymmetric solutions such that A_{-t} is the reflection of A_t, $H \sim et^2$ and $R \sim ft^2$, and $f > 0$, $ced < 0$, and there are no other nearby periodic solutions which are small deformations of the fold solution.

SB2 (subcritical symmetry breaking): The same as SB1 but with $f < 0$, $ced > 0$.

Note that if the conditions of reversibility of the system and symmetry of the fold solution are dropped then SC becomes the sole generic possibility (Meyer 1970).

If we assume (assumption 2 of §1) that a fold point is generic in the above sense, then one of the above three possibilities must occur. We will find in the next section, numerically, a fold point for the uniformly travelling water-wave problem which is a turning point (suitably interpreted) in a family of symmetric periodic solutions,
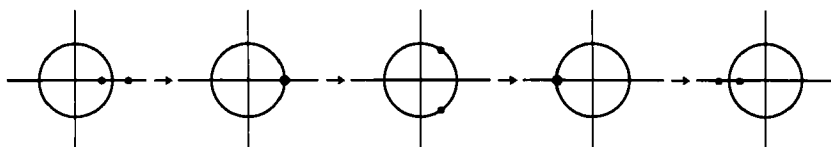
FIGURE 1. The typical behaviour of the Floquet multipliers of a curve of periodic orbits of a two-degrees-of-freedom Hamiltonian system.

and hence cannot be in case SB1 or SB2. Thus, under assumption 2 it must be in case SC.

Let us examine two of the primary consequences of SC. A periodic solution can bifurcate from a given one only if the latter has a Floquet multiplier at a root of unity (an application of the implicit function theorem). Thus bifurcations can happen only if there is a residue $0 \leqslant R \leqslant 1$. Typically, there is a bifurcation of type (p, q) every time a pair of Floquet multipliers pass through $\exp(\pm 2\pi i p/q)$, that is every time a residue passes through $\sin^2 \pi p/q$ (Meyer 1970; Rimmer 1983). Hence on one side of a saddle-centre bifurcation ($R > 0$) we expect a dense set of bifurcations to occur in an interval ($0 \leqslant R \leqslant 1$), while the normal Floquet multipliers traverse the unit circle (figure 1). The periodic solution is said to be elliptic for $0 < R < 1$, regular hyperbolic for $R < 0$, and inverse hyperbolic for $R > 1$. Secondly, there is no symmetry breaking at a saddle-centre bifurcation. These consequences of a saddle-centre bifurcation agree perfectly with the observed bifurcation results previously reported for water waves. So to explain the results, all we have to do, subject to assumptions 1 and 2, is to demonstrate the existence of a fold in the set of symmetric periodic orbits.

For the limit of the flat wave ($A = 0$), one can easily show that $+1$ is the only multiplier and has multiplicity 2, though for finite depth h one obtains the additional multipliers $\mu = e^{\kappa\lambda}$, for all solutions κ of $c^2\kappa = g \tan \kappa h$, where λ is the solution of $2\pi c^2/\lambda = g \tanh 2\pi h/\lambda (c^2 < gh)$. So for small A, the dynamics is essentially that of a one degree of freedom Hamiltonian system and no bifurcation can happen. The possibility of bifurcation from the Stokes wave requires the existence of a second pair of Floquet multipliers lying on the unit circle. We will find this at larger amplitude in the next section.

Note that the same behaviour of the multipliers was obtained by Zufiria (1987a) in a two degree of freedom Hamiltonian system resulting from a truncation of the full system for small-amplitude shallow water waves. Furthermore, Zufiria observed qualitatively similar bifurcation diagrams for both the truncated model and the full system for infinite depth (Zufiria 1987b), supporting the validity of our assumptions.

5. Existence of a fold

We now present numerical evidence for a fold at A_0.

Diagrams showing quantities such as the velocity c or the mean energy per unit horizontal length E in the rest frame against the amplitude A of the Stokes wave do not show any turning points at A_0 (Chen & Saffman 1980; Longuet-Higgins & Fox 1978). In our case, the Hamiltonian is the flow-force invariant $\bar{S}$, but it does not show a turning point either (as we shall show in figure 3). This might appear to contradict the idea of a fold.

However, what is actually done in most numerics on uniformly travelling water waves is to follow a path of waves of fixed period, allowing the wave speed c to vary. The velocity c should be regarded as an external parameter to the dynamical system
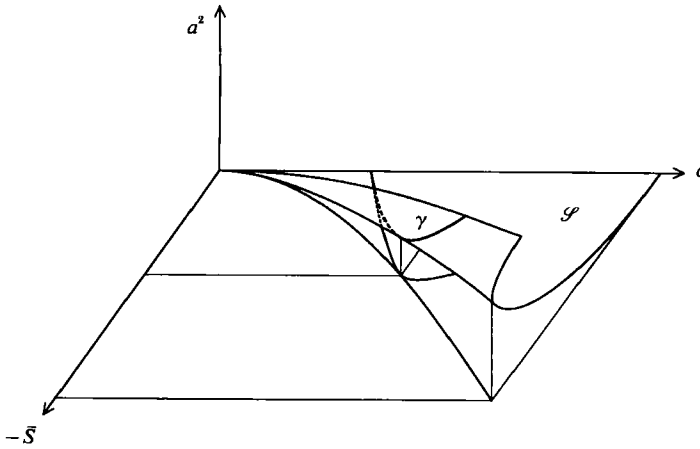
FIGURE 2. Conjectured picture of the surface $\mathscr{S}$ of Stokes waves over the $(c, \bar{S})$-plane, showing a fold, the path γ of waves of a fixed wavelength, and their projections onto the $(c, \bar{S})$-plane.

(entering via the boundary conditions). The definition of a fold requires finding nearby solutions with the same energy to first order and the same value of all external parameters, for close to but not necessarily the same period. Hence we need to convert the results for fixed wavelength to results for fixed speed.

We will obtain a better understanding of the bifurcation problem if we represent the solutions over a higher-dimensional space, namely the space $(c, \bar{S})$. We represent Stokes' family of periodic solutions as a surface $\mathscr{S}$ in a three-dimensional bifurcation diagram, taking a^2 as a measure of the size of the waves and $(c, \bar{S})$ as parameter plane. Note that the choice of c and $\bar{S}$ is crucial; we do not get the same results with c and E (see §8).

We claim that the surface $\mathscr{S}$ has a fold in its projection to $(c, \bar{S})$, and the path of Stokes' waves of given wavelength λ pass over the fold precisely at A_0 (figure 2).

Since we are in the case of infinite depth, this is easy to verify numerically because of the scaling symmetry. We recall the symmetry properties of the problem of uniformly travelling water waves on fluid of infinite depth (see Benjamin & Olver (1982) for a full report of these properties). If $[c, \eta(x), \psi(x, y)]$ defines a uniformly travelling water wave, then so do the following:

$$[c, \eta(x-\delta)+\epsilon, \psi(x-\delta, y-\epsilon)], \quad \forall \delta, \epsilon \in \mathbb{R} \text{ (translation symmetry)},$$

$$[c, \eta(-x), \psi(-x, y)] \text{ (reflection symmetry)},$$

$$[\beta c, \beta^2 \eta(x/\beta^2), \beta^3 \psi(x/\beta^2, y/\beta^2)], \quad \forall \beta \in \mathbb{R} \backslash \{0\} \text{ (scaling symmetry)}.$$

From the definition of $\bar{S}$ and the scaling symmetry, we see that if a wave exists for parameter values $(c, \bar{S})$ then a similar rescaled wave exists for parameter values $(\beta c, \beta^4 \bar{S})$, for all $\beta > 0$.

It follows that if the surface $\mathscr{S}$ has a fold curve, then it is of the form $\bar{S} = \mu c^4$, for some $\mu \in \mathbb{R}$. Conversely, if the projection of a path of periodic waves onto the $(c, \bar{S})$-plane is tangent somewhere to one of the family of quartics $\bar{S} = \mu c^4$, then that quartic is a fold curve for $\mathscr{S}$. For a generic path going over a fold, the projections of the path and the fold onto the parameter plane meet tangentially, with second-order contact
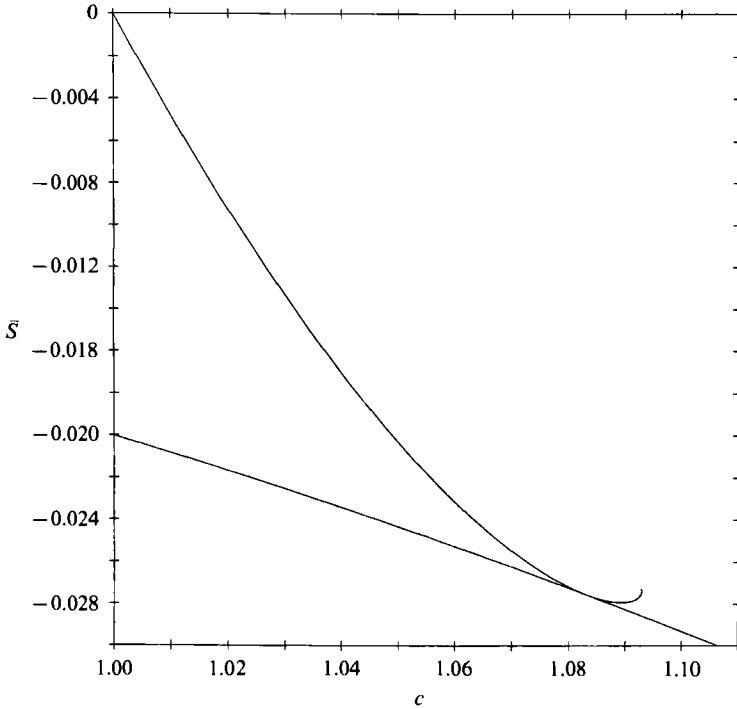
FIGURE 3. Computed path n the $(c, \bar{S})$-plane for waves of wavelength 2π, showing tangency to a quartic $\bar{S} = \mu c^4$ at $A \approx 0.40222$ ($c \approx 1.08309$).

(see figure 2). Hence we have a simple test for a path to go over a fold, namely $\bar{S}'/\bar{S} = 4c'/c$, where the prime refers to the derivative along the path.

We used this criterion to search for fold points in the Stokes family. We computed the Stokes family for fixed wavelength $\lambda = 2\pi$ in the form of a Fourier expansion of the coordinates x and y in terms of the conjugate variables ϕ and ψ (hodograph transform):

$$x + iy = -(\phi + i\psi)/c + \tfrac{1}{2}i(c^2 + a_0) + i\sum_{k=1}^{\infty} a_k e^{ik(\phi + i\psi)/c}.$$

Note that this differs by the amount $\tfrac{1}{2}ic^2$ from the representation used in Longuet-Higgins (1984a), which reflects our different choice of Bernoulli constant and hence mean level. The Fourier coefficients and the velocity c are computed by solving the Longuet-Higgins system of quadratic equations (Longuet-Higgins 1984a), for equally spaced values of the apparently monotonic parameter

$$B = \sum_{k=1}^{\infty} ka_k$$

in the range $[0, 6.0]$, which corresponds to A in the range $[0, 0.43775]$. The advantage of B as parameter is that it is infinite for the limiting wave, and equal steps in B seem to lead to controllable changes in the wave, in contrast to A. We truncated at some order N, and used the initial approximation below to get started:

$$a_0 = -1 - 2B^2, \quad a_1 = B, \quad a_2 = B^2, \quad c^2 = 1 + B^2.$$

In parallel, we computed $\bar{S}$ using the Fourier coefficients $\{a_k\}$; see the Appendix for the calculation of $\bar{S}$ in terms of (ϕ, ψ)-coordinates. A check on the accuracy of the
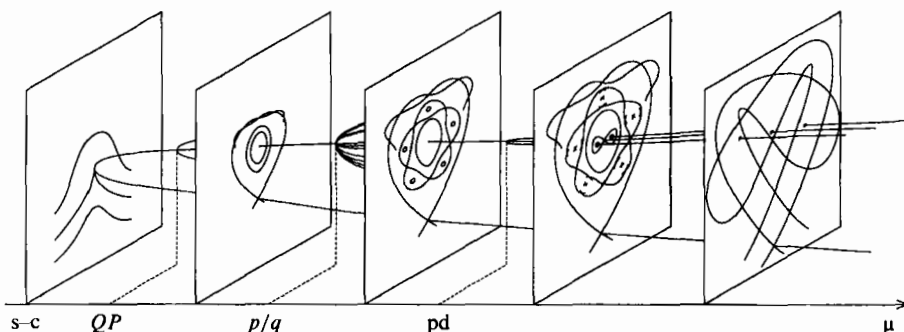
FIGURE 4. Conjectured sequence of Poincaré maps on crossing the fold, indicating spatially quasiperiodic, periodic, heteroclinic and chaotic waves coexisting with the Stokes family. s-c, indicates the saddle-centre bifurcation; pd, the period doubling bifurcation; QP, one of a dense set of bifurcation points between s-c and pd of a circle of quasi-periodic solutions; p/q, one of a dense set of bifurcations between s-c and pd of periodic solutions. Note the formation of transverse homoclinic points and resulting chaos, which eventually becomes a complete horseshoe as shown in the final section.

wave computation, which we used to determine how many Fourier coefficients to keep, was to use the theoretical invariance of $\bar{S}$, computing it at both the trough and crest and imposing a tolerance on the difference $\bar{S}_{\mathrm{err}}$ in the values obtained.

Figure 3 shows a plot in the $(c, \bar{S})$-plane of the results obtained for B in the above range. We found a tangency to the family of quartics at $A \approx 0.402\,22$ with $N = 198$ and $\bar{S}_{\mathrm{err}} \approx 10^{-10}$. This corresponds to within numerical error to the previously quoted value of A_0. Thus we claim that A_0 is a fold point. Furthermore, the tangency was found numerically to be of second order, thus verifying the condition $b \neq 0$ of SC, and in particular ruling out SB1 and SB2. Hence we explain the observed dense interval of bifurcations and absence of symmetry breaking at A_0.

6. Predictions

The dynamical systems viewpoint suggests the occurrence of many more types of uniformly travelling wave with $(c, \bar{S})$ close to the fold. The generic behaviour of a reversible Hamiltonian system (or family of them) near a saddle-centre is indicated in figure 4 (cf. the area-preserving Hénon map (Hénon 1969)). Here we show sketches of Poincaré sections at several parameter values $\mu = \bar{S}/c^4$ on passing a saddle-centre. A Poincaré section in our context is the restriction to fixed $(c, \bar{S})$ of a transverse four-dimensional section Σ to the saddle-centre orbit in the extended phase space where c is regarded as an additional coordinate. The Poincaré map is the first return map of this section to itself. What is sketched is part of the behaviour of the Poincaré map for a sequence of parameter values on crossing the fold.

What generically occurs is a complicated story: besides bifurcation of periodic orbits, there are bifurcations of quasi-periodic orbits, heteroclinic orbits and chaotic ones. Furthermore, many phenomena have been omitted from our sketch (e.g. collapse at $\frac{1}{3}$ resonance). We summarize the main features (for a review see MacKay 1987):

1. A saddle and centre (elliptic orbit) which meet parabolically in a saddle-centre.
2. Bifurcations of type p/q, $0 < p/q \leqslant \frac{1}{2}$, p, q coprime, from the elliptic fixed point, roughly half of the resulting branches being themselves elliptic initially.

3. Bifurcations of further periodic orbits from each branch of elliptic periodic orbits.

4. Bifurcations of asymmetric orbits (e.g. period 6 in the Hénon map; MacKay 1982 *b*).

5. Saddle-centre bifurcations of other periodic orbits not attached to the principal bifurcation tree (e.g. period 5 in the Hénon map; MacKay 1982 *a*).

6. Bifurcation of circles of quasi-periodic orbits from each branch of elliptic periodic orbits, which later break into 'cantori'.

7. At least four heteroclinic orbits associated with each p/q-bifurcation ($q \neq 1, 2$), generically transverse.

8. Chaotic orbits associated with horseshoes generated by transverse heteroclinic orbits.

9. Eventual formation of a complete horseshoe as the maximal invariant set locally.

Points 1 and 2 we have already discussed in §4. Some other aspects of this picture have also been verified. In particular, on the suggestion of one of us, Zufiria (1987 *b*) followed many branches of the bifurcation tree (point 3), and found agreement with that for the Hénon map, including a symmetry breaking bifurcation at period 6 (point 4). As far as we know, the other aspects mentioned above have not yet been looked for. We are developing a scheme to find quasi-periodic waves numerically. Given a method for computing general bounded uniformly travelling waves, one could draw 'surface of section' plots, e.g. plotting (η, w) every time $F_{Y0} = 0$ with $F_{YY0} > 0$, and compare with figure 4.

Similar conclusions should hold for finite depth, though the scaling symmetry is then lost. Also if surface tension is added, we except an analogous picture with all the generic features of three-degrees-of-freedom Hamiltonian systems, since surface tension permits one more oscillatory degree of freedom.

7. Conjectures about the approach to the highest wave

Having provided some insight into the bifurcations in the range $[0, A_{\frac{1}{2}}]$, an interesting question is the bifurcation behaviour when A increases well beyond $A_{\frac{1}{2}}$. In particular, what happens in the limit $A \to A_{\max}$? From the asymptotic analysis of Longuet-Higgins & Fox (1978), we know that both E and c converge oscillatorily (but not in phase) to limiting values as A approaches $A_{\max}$. We expect the same for $\bar{S}$. Thus we expect the family of Stokes waves of fixed wavelength to give a spiral shaped projection into the $(c, \bar{S})$-plane. Indeed one can see the beginnings of such a spiral in figure 3. Hence there should be an infinite sequence of fold points (saddle-centre bifurcations), two for each full revolution along the spiral, with a pair of Floquet multipliers going through $+1$ at such points (figure 5). The question is in which direction do the Floquet multipliers move on passing round successive folds? Is there just one pair (α, α^{-1}) for which the residue oscillates repeatedly through 0, or is a new pair involved at each fold point, for which the residue passes there from negative to positive? We are developing numerical methods to look at this question. Possibly the analysis of Maddocks (1987) could also provide an answer.

Also an interesting theoretical approach to the highest and almost highest waves comes out of our Hamiltonian formulation. Our symplectic form on M' becomes degenerate wherever $U_0 = 0$. This is the case at the crest of the highest wave. Thus the Hamiltonian vector field is not defined there. We are investigating the generic behaviour of finite-dimensional Hamiltonian systems near degeneracies of the
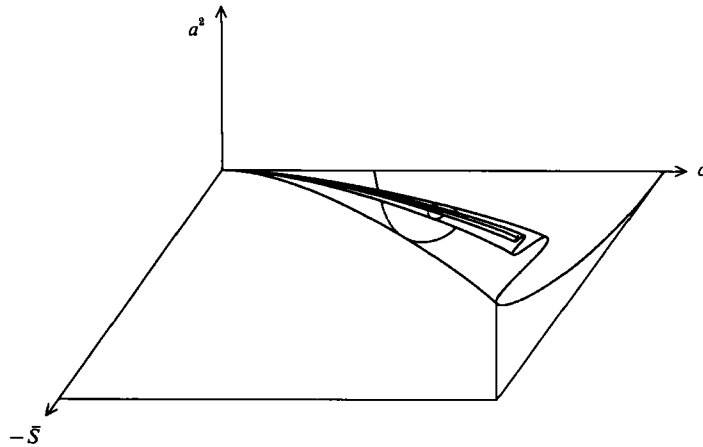
FIGURE 5. Conjectured continuation of $\mathcal{S}$ to the Stokes waves of highest amplitude, showing the conjectured form of the path of waves of fixed wavelength.

symplectic form, in order to see whether this will shed some light on the behaviour of the highest and near highest waves. Preliminary investigations, however, suggest that ker ω has dimension 1 at the highest wave, whereas in finite dimensions it must always be even. Hence it is likely that the highest wave is an essentially infinite-dimensional problem.

8. Conclusion

We have found a Hamiltonian formulation for uniformly travelling irrotational gravity waves on ideal fluid of infinite depth, and numerically demonstrated the existence of a fold in Stokes' family of periodic waves. This allows us to interpret and understand many numerical results on bifurcation of such waves as generic behaviour for a Hamiltonian system, subject to assumption 1 of the existence of an appropriate centre manifold. This assumption is likely to be technically difficult to justify, but we feel it is conceptually quite natural. Assumption 2 of a generic fold is in principal testable: 'all' one has to do is prove the existence of the fold solution and compute the relevant combinations of Taylor coefficients with sufficient accuracy to be sure that they are non-zero. Our viewpoint also suggests many interesting predictions for existence of quasi-periodic, heteroclinic and chaotic uniformly travelling waves.

Our Hamiltonian formulation may well generalize to many other uniformly travelling-wave problems, for example, with density stratification and/or vorticity. An interesting question is whether all wave systems which have a Hamiltonian formulation in time also have a Hamiltonian formulation in space for their uniformly travelling waves. In the light of the results of Benjamin (1984) and the present paper, the Hamiltonian is likely to be Benjamin's flow-force invariant in every case, but the question is to determine the correct phase space and symplectic form. A recent preprint of Bridges (1991) sheds light on this.

We conclude with some comments on Tanaka's instability and time-dependence.

Our fold points, turning points in $\bar{S}/c^4$, appear to be distinct from those in E (relevant to temporal stability to superharmonic disturbances; Tanaka 1985; Saffman 1985) and in c, though we expect them to be interleaved. The quantity E is

the energy per unit horizontal length relative to the state of no flow. Using our conventions:

$$E = \left\langle \tfrac{1}{2}\eta^2 + \tfrac{1}{2}\int_{-\infty}^{\eta} [(\psi_y + c)^2 + \psi_x^2]\,\mathrm{d}y_x \right\rangle.$$

For its evaluation in terms of the Fourier coefficients $\{a_k\}$ see Longuet-Higgins (1984b). It reaches its first turning point (for waves of fixed wavelength) at $A_E \approx 0.4292$, and the first turning point in c is at $A_c \approx 0.4359$ (Longuet-Higgins 1985). We found the first turning point in E/c^4 to be at $A \approx 0.42294$ (with $N = 288$ and $\bar{S}_{\mathrm{err}} \approx 1.1 \times 10^{-8}$), which is in between $A_{\frac{1}{2}}$ and A_E. In particular, it has nothing to do with A_0. We emphasize the importance of the role of $\bar{S}$ in understanding the set of uniformly travelling water waves.

Finally, uniformly travelling solutions, even though unstable (e.g. by the Benjamin–Feir (1967) instability), could be useful in helping to understand the full time-dependent problem, because they and their stable and unstable manifolds form a skeleton along which it is possible that a lot of the dynamics moves (cf. Eckmann & Procaccia 1990).

In memory of Charles Amick.

Appendix. Benjamin's flow-force invariant

In Benjamin's original definition (Benjamin 1984), the flow force invariant $\bar{S}$ is given as an integral over y of functions of x and y. We first generalize his definition to allow any path of integration from the bottom to the surface:

$$\bar{S} = -\tfrac{1}{2}\eta^2(x) + \int_{y \to -\infty}^{x,\,\eta(x)} \tfrac{1}{2}((\psi_y + c)^2 - \psi_x^2)\,\mathrm{d}y + \psi_x(\psi_y + c)\,\mathrm{d}x,$$

where the integrand is now an exact differential, giving the same result as before. Then, using the hodograph transform, $\bar{S}$ can be expressed in terms of ϕ and ψ as independent variables, as follows:

$$\bar{S} = -\tfrac{1}{2}y^2(\phi,0) + \int_{\psi \to \infty}^{\phi,\,0} (\tfrac{1}{2}y_\psi(Q^2 + c^2) + c)\,\mathrm{d}\psi + \tfrac{1}{2}x_\psi(Q^2 - c^2)\,\mathrm{d}\phi,$$

where $Q^2 = (x_\psi^2 + y_\psi^2)^{-1}$. To simplify the computation of $\bar{S}$ from the Fourier expansion of the Stokes wave, we chose the integration paths $\phi = 0$ (to a crest) and $\phi = \pi$ (to a trough), for which $x_\psi = 0$.

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